Lecture 16

Friday, June 3, 2022 10:35 AM

* Prayer

* Spiritual thought

<u>Recall</u>: change of variables

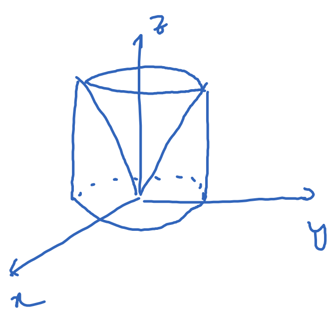
$$\iint_R f(x,y)\,dA \xrightarrow{(x,y)\to(u,v)} \iint_{R'} f(x(u,v),y(u,v))\,J\,dA$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$\iiint_R f(x,y,z)\,dV \xrightarrow{(x,y,z)\to(u,v,w)} \iiint_{R'} f(x(u,v,w),y(u,v,w),z(u,v,w))\,J\,dV$$

$$J = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$$

<u>Ex</u>



R = the solid bounded by the cone $z=\sqrt{x^2+y^2}$, the cylinder $x^2+y^2=1$, and the plane $z=0$.

Find $\iiint_E z\,dV$.

Cylindrical coords:

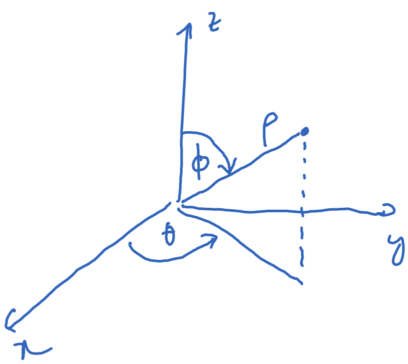
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$R': \quad \begin{aligned} &0 \le r \le 1 \\ &0 \le \theta \le 2\pi \\ &0 \le z \le r \end{aligned}$$

$$\iiint_R z\,dV = \iiint_{R'} z\,J\,dV = \iiint_{R'} zr\,dV = \int_0^{2\pi}\int_0^1\int_0^r zr\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^1 \frac{r^3}{2}\,dr = \frac{\pi}{4}.$$

Spherical coordinates



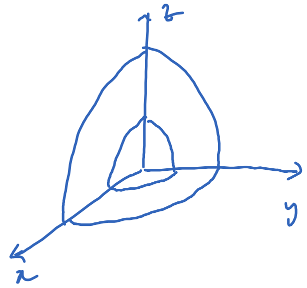
$$(x,y,z) \longrightarrow \underbrace{(\rho,\theta,\phi)}_{\text{spherical coordinates}}$$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{aligned} &0 \le \theta \le 2\pi, \\ &0 \le \phi \le \pi \end{aligned}$$

$$\boxed{J = \rho^2\sin\phi}$$

Ex $(x,y,z) = (0,1,\sqrt{3})$. Find ρ, θ, ϕ.

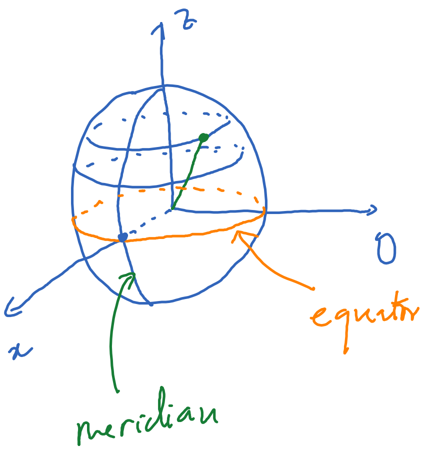
Ex



$$\left.\begin{array}{l} 1 \le \rho \le 2 \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{array}\right\} R'$$

$$\iiint_R x\, dV = \iiint_{R'} \rho \sin\phi \cos\theta\, \rho^2 \sin\phi\, dV = \int_1^2 \int_0^{\pi/2} \int_0^{\pi/2} \rho^3 \sin^2\phi \cos\theta\, d\theta\, d\phi\, d\rho$$

$$= \cdots$$



Fix θ, move ϕ : longitude $= \theta$

Fix ϕ, move θ : latitude $= 90^\circ - \phi$

Use the command TransformedRegion in Mathematica to draw the region described by spherical coordinates :

$$1 \le \rho \le 2, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le \frac{\pi}{2}$$

Mathematica :

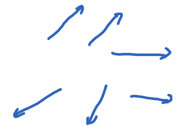
R = ImplicitRegion[$1 \le \rho \le 2$ && $0 \le \theta \le \pi$ && $0 \le \phi \le \frac{\pi}{2}$, {ρ, θ, φ}]

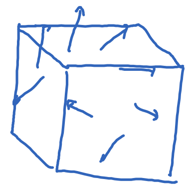
f[ρ_, θ_, φ_] := {ρ sinφ cosθ, ρ sinφ sinθ, ρ cosφ}

R′ = TransformedRegion[R, f]

Region[R′]

Vector fields

2D vector fields:  $f(x,y) = (x+y, x^2y)$

3D vector field:  $f(x,y,z) = (xz, y^2, z+y)$

A 2D vector field is a function $f: \mathbb{R}^2 \to \mathbb{R}^2$.

A 3D vector field is a function $f: \mathbb{R}^3 \to \mathbb{R}^3$.

Visually, a vector field is a map of arrows. It can be visualized on Mathematica using the command VectorPlot:

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VectorPlot[{x+y, x-y}, {x,-1,1}, {y,-1,1}]
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Gradient is a vector field.

Ex $\quad f(x,y) = x^2+y^2$

$$\nabla f(x,y) = (2x, 2y)$$

Ex $\quad f(x,y,z) = x^2+y^2+z^2$

$$\nabla f(x,y,z) = (2x, 2y, 2z)$$

Next time: line integral.